MA 441, Fall 2010

Assignment 4.

This homework is extended to *Thursday* 10/21/2010.

There are total 47 points in this assignment. 40 points is considered 100%. If you go over 40 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

This assignment covers sections 4.1–4.2 and a bit of 4.3.

(1) (Theorem 4.1.2) Let $A \subseteq \mathbb{R}$. Prove that
  (a) [3pt] If a number $c \in \mathbb{R}$ is a cluster point of A, then there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.
  (b) [3pt] If there exists a sequence (a_n) in A such that $\lim(a_n) = c \in \mathbb{R}$ and $a_n \neq c$ for all $n \in \mathbb{N}$, then $c \in \mathbb{R}$ is a cluster point of A.

(2) (Exercises 4.1.1,2) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)
  (a) [2pt] $|x^2 - 1| < 1/2$, $c = 1$.
  (b) [2pt] $|x^2 - 1| < 10^{-3}$, $c = 1$.
  (c) [2pt] $|x^2 - 1| < \frac{1}{10^{-3}}$, $c = 1$.
  (d) [3pt] $|x + x^2 + 1/x - 6.5| < 1/2$, $c = 2$.
  (e) [3pt] $|x^3 \sin x - 0| < 0.001$, $c = 0$.

(3) (Exercise 4.1.11) Show that the following limits do *not* exist in $\mathbb{R}$.
  (a) [2pt] $\lim\limits_{x \to 0} \frac{1}{x^2}$.
  (b) [2pt] $\lim\limits_{x \to 0} (x + \operatorname{sgn}(x))$.

(4) (Exercise 4.1.14) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.
  (a) [3pt] Show that f has limit at $x = 0$ (*Hint*: you can use squeeze theorem).
  (b) [3pt] Prove that if $c \neq 0$, then f does not have limit at c. (*Hint*: you can use sequential criterion.)

— *see next page* —

(5) [3pt] (Theorem 4.2.4 for difference) Using sequential criterion, prove that limit of functions preserves difference. That is, prove the following:
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A, and f, g be functions on A to $\mathbb{R}$. If $\lim_{x\to c} f = L$, and $\lim_{x\to c} g = M$, then $\lim_{x\to c} f - g = L - M$.

(6) (Exercise 4.2.1,2) Using arithmetic properties of limit, find the following limits.
   (a) [2pt] $\lim_{x\to 1} \frac{x^2+2}{x^2-2}$.
   (b) [2pt] $\lim_{x\to 2} \left(\frac{1}{x+1} - \frac{1}{2x}\right)$.
   (c) [2pt] $\lim_{x\to 0} \frac{(x+1)^2-1}{x}$.
   (d) [2pt] $\lim_{x\to 0} \frac{(x^2+1)^2-1}{x}$.
   (e) [2pt] $\lim_{x\to \infty} \frac{(x+1)^2-1}{x}$.
   (f) [2pt] $\lim_{x\to \infty} \frac{(x+1)^2-1}{x^2}$.

(7) [4pt] (Exercise 4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x\to c} g = 0$. Prove that $\lim_{x\to c} fg = 0$.